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SHORTER COMMUNICATIONS

LAMINAR HEAT TRANSFER IN A ROUND TUBE WITH VARIABLE CIRCUMFERENTIAL OR ARBITRARY WALL HEAT FLUX

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NOMENCLATURE

a, radius of tube;
a_m, b_m, coefficients of harmonic terms in Fourier series expansion;
d, diameter of tube;
K, thermal conductivity;
Nu, local Nusselt number $= qd/K(t_w - t_m)$;
Pe, Péclét number $= \bar{u}d/\alpha$;
q, heat flux at the wall;
$\bar{q}$, average heat flux at the wall;
r, radial co-ordinate;
r^*, non-dimensional radial distance $= r/a$;
Re, Reynolds number $= \bar{u}d/\nu$;
t, temperature of fluid;
t^*, non-dimensional temperature $= K(t - t_i)/\bar{q}a$;
t_i, t_w, inlet and wall temperature;
t_m, bulk mean temperature;
$\bar{u}, u$, average and axial component of velocity;
x, axial co-ordinate;
X, non-dimensional axial distance $= \alpha x/2\bar{u}a^2$;
α, thermal diffusivity;
ν, kinematic viscosity;
θ, circumferential co-ordinate.

INTRODUCTION

A LIMITED number of papers are available recently to study the asymptotic behaviour of temperature solution for flow through channels having variable heat flux at the boundary wall. Reynolds [1] found the effect of variable circumferential heat flux to be quite significant for turbulent flow in a circular tube. Sutherland and Kays [2] investigated similar problem for annulus geometry considering both laminar and turbulent flow. This again was solved only for fully developed velocity and temperature profile. An experimental investigation on turbulent heat transfer in a circular tube with variable circumferential heat flux was reported by Black and Sparrow [3]. The temperature solutions in thermal entrance region for slug and laminar flow through a tube were obtained by Hsu [4] for axially varying sinusoidal heat flux by applying Duhamel's superposition theorem to the entrance region solution for a uniform wall heat flux.

In this brief note the authors present first the temperature solution in the thermal entrance region for developed laminar flow in a tube with variable circumferential wall heat flux. Secondly, the temperature for arbitrary wall heat flux in a tube of finite length is obtained as an extension of the previous solution.

FIRST PROBLEM: VARIABLE CIRCUMFERENTIAL HEAT FLUX

Neglecting axial conduction and viscous dissipation the energy equation for incompressible developed laminar flow in a tube with constant fluid properties may be written as

$$\alpha\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2}\right) = 2\bar{u}\left(1 - \frac{r^2}{a^2}\right)\frac{\partial t}{\partial x}. \tag{1}$$

The boundary and initial conditions for equation (1) are taken as

$$r = a: \quad K\frac{\partial t}{\partial r} = \bar{q}[1 + \sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta)];$$

$$x = 0: \quad t = t_i.$$

The dimensionless form of the above equations are expressed as

$$\frac{\partial^2 t^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial t^*}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 t^*}{\partial \theta^2} = (1 - r^{*2})\frac{\partial t^*}{\partial X} \tag{2}$$

$$r^* = 1: \quad \frac{\partial t^*}{\partial r^*} = 1 + \sum_{m=1}^{\infty}(a_m \cos m\theta + b_m \sin m\theta);$$

$$X = 0: \quad t^* = 0.$$

For the solution of equation (2) let

$$t^* = t_1(X) + t_2(r^*) + t_3(r^*, \theta) + t_4(X, r^*, \theta). \tag{3}$$

Substituting t^* in equation (2) the resulting equations can be written as

$$\frac{d^2 t_2}{dr^{*2}} + \frac{1}{r^*}\frac{dt_2}{dr^*} = (1 - r^{*2})\frac{dt_1}{dX} \tag{4a}$$

$$\frac{\partial^2 t_3}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial t_3}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 t_3}{\partial \theta^2} = 0 \tag{4b}$$

$$\frac{\partial^2 t_4}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial t_4}{\partial r^*} + \frac{1}{r^{*2}}\frac{\partial^2 t_4}{\partial \theta^2} = (1 - r^{*2})\frac{\partial t_4}{\partial X} \tag{4c}$$

with the boundary and initial conditions

$$r^* = 1; \quad \frac{\mathrm{d}t_2}{\mathrm{d}r^*} = 1, \quad \frac{\partial t_3}{\partial r^*} = \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta),$$

$$\frac{\partial t_4}{\partial r^*} = 0; \quad X = 0: \quad t_4 = -(t_1 + t_2 + t_3).$$

Equation (4a) can be split up into two components and then solved for t_1 and t_2. Equation (4b) can be solved by the method of separation of variables. The final solutions are

$$t_1 = 4X + D \tag{5a}$$

$$t_2 = r^{*2} - \frac{r^{*4}}{4} + F \tag{5b}$$

$$t_3 = \sum_{m=1}^{\infty} \frac{r^{*m}}{m}(a_m \cos m\theta + b_m \sin m\theta) \tag{5c}$$

in which D and F are arbitrary constants.

The solution of equation (4c) satisfying the boundary and initial condition may be obtained by the technique of separation of variables and expressed as

$$t_4 = c_0 - \sum_{s=1}^{\infty} c_{0s}R_{0s}\,\mathrm{e}^{-\beta_{0s}^2 X}$$
$$- \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} c_{ms}R_{ms}(a_m \cos m\theta + b_m \sin m\theta)\,\mathrm{e}^{-\beta_{ms}^2 X}. \tag{6}$$

The eigenfunctions R_{ms} (r^*, β_{ms}) satisfy the differential equation

$$r^{*2}R''_{ms} + r^*R'_{ms} - (m^2 - \beta_{ms}^2 r^{*2} + \beta_{ms}^2 r^{*4})R_{ms} = 0 \tag{7}$$

in which the derivatives are with respect to r^*. It may be noted that for $m = 0$, the above leads to an equation solved earlier by Siegel *et al.* [5] for constant heat flux at the boundary. The eigenvalue equation is obtained by satisfying the boundary condition for t_4 so that

$$R'_{ms}(1, \beta_{ms}) = 0. \tag{8}$$

Combining equations (5a), (5b), (5c) and (6) the final solution of the problem may be expressed as

$$t^* = 4X + r^{*2} - \frac{r^{*4}}{4} + A + \sum_{m=1}^{\infty} \frac{r^{*m}}{m}(a_m \cos m\theta$$
$$+ b_m \sin m\theta) - \sum_{s=1}^{\infty} c_{0s}R_{0s}\,\mathrm{e}^{-\beta_{0s}^2 X} - \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} c_{ms}R_{ms}$$
$$\times (a_m \cos m\theta + b_m \sin m\theta)\,\mathrm{e}^{-\beta_{ms}^2 X} \tag{9}$$

It has been proved in [7] that to satisfy the initial condition for t_4 we must have

$$A = -\frac{7}{24}, \quad c_{ms} = \frac{2}{\beta_{ms} \left.\dfrac{\partial^2 R_{ms}(r^*, \beta)}{\partial r^* \partial \beta}\right|_{\beta = \beta_{ms}}^{r^*=1}} \quad (\text{for } m \geqslant 0)$$

The non-dimensional wall temperature is obtained from the temperature solution (9) by substituting $r^* = 1$. This may be written as

$$t_w^* = 4X + \frac{11}{24} + \sum_{m=1}^{\infty} \frac{1}{m}(a_m \cos m\theta + b_m \sin m\theta)$$
$$- \sum_{s=1}^{\infty} c_{0s}R_{0s}(1, \beta_{0s})\,\mathrm{e}^{-\beta_{0s}^2 X} - \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} c_{ms}R_{ms}(1, \beta_{ms})$$
$$(a_m \cos m\theta + b_m \sin m\theta)\,\mathrm{e}^{-\beta_{ms}^2 X}. \tag{10}$$

The expression for local Nusselt number based on difference of wall temperature and bulk mean temperature is found to be

$$Nu = \frac{2[1 + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta)]}{\frac{11}{24} + \sum_{m=1}^{\infty} \frac{1}{m}(a_m \cos m\theta + b_m \sin m\theta) - \sum_{s=1}^{\infty} c_{0s}R_{0s}(1, \beta_{0s})\,\mathrm{e}^{-\beta_{0s}^2 X} - \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} c_{ms}R_{ms}(1, \beta_{ms})(a_m \cos m\theta + b_m \sin m\theta)\,\mathrm{e}^{-\beta_{ms}^2 X}}. \tag{11}$$

To evaluate wall temperature or Nusselt number from equation (10) or (11) up to fifth harmonic terms β_{ms} and $c_{ms}R_{ms}(1, \beta_{ms})$ were calculated by a 7044 IBM digital computer and are reported in Tables 1 and 2. The higher eigenvalues and the corresponding coefficients may be evaluated within one per cent accuracy by W.K.B. approximation method due to Sellars *et al.* [6] and are given by

$$\beta_{ms} = 4s + 2m + \tfrac{4}{3}, \quad c_{ms}R_{ms}(1, \beta_{ms}) = 2{\cdot}40\,1429\,\beta_{ms}^{-\frac{4}{3}}.$$

The derivation of the two equations above is detailed in [7].

Table 1. Eigenvalues of equation (8)

s	β_{0s}	β_{1s}	β_{2s}	β_{3s}	β_{4s}	β_{5s}
0	0·0000	2·8846	5·0676	7·2302	9·3792	11·5099
1	5·0675	7·1183	9·1576	11·2076	13·2723	15·3490
2	9·1576	11·1789	13·1972	15·2212	17·2549	19·2994
3	13·1972	15·2093	17·2202	19·2344	21·2546	23·2828
4	17·2202	19·2282	21·2355	23·2448	25·2585	27·2774
5	21·2355	23·2412	25·2466	27·2537	29·2584	31·2931

Table 2. Values of $c_{ms}\,R_{ms}\,(1, \beta_{ms})$

s	$c_{0s}R_{0s}$	$c_{1s}R_{1s}$	$c_{2s}R_{2s}$	$c_{3s}R_{3s}$	$c_{4s}R_{4s}$	$c_{5s}R_{5s}$
0	—	0·6699	0·2354	0·1108	0·0585	0·0328
1	0·1987	0·1108	0·0727	0·0510	0·0364	0·0269
2	0·0692	0·0492	0·0374	0·0293	0·0236	0·0193
3	0·0365	0·0287	0·0233	0·0194	0·0169	0·0158
4	0·0230	0·0191	0·0162	0·0139	0·0112	0·0120
5	0·0160	0·0137	0·0119	0·0104	0·0099	0·0079

Without considering any general peripheral heat flux, to study the effects of first and second harmonic heat fluxes separately compared to constant heat flux case a variation of 20 per cent over average heat flux is considered in each of the following examples:

$$\text{case (i)}\ q = \bar{q}(1 + 0{\cdot}2\cos\theta)$$

$$\text{case (ii)}\ q = \bar{q}(1 + 0{\cdot}2\cos 2\theta).$$

The non-dimensional wall temperatures and Nusselt numbers are plotted graphically for these two particular cases in Figs. 1–3. The following effects of the two harmonic heat fluxes are observed from these curves:

(i) The wall temperature is high where the heat flux is greater than the average heat flux and vice-versa.
(ii) The first harmonic heat flux has a much greater effect on the wall temperature and Nusselt number than the second harmonic heat flux other conditions remaining unaltered.

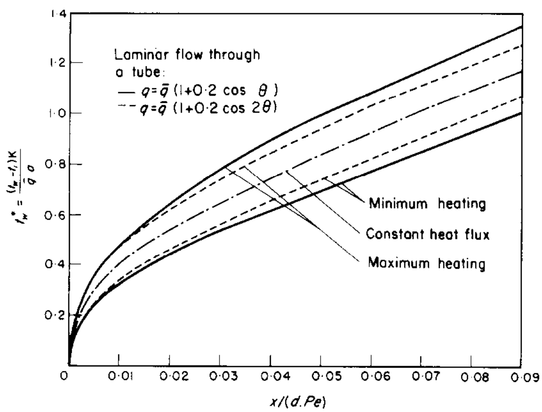

FIG. 1. Wall temperature vs. axial distance for laminar flow through a tube.

It may be understood from equations (10) and (11) by examining the coefficients of harmonic terms with the help of Tables 1 and 2 that for same amplitude the first harmonic has greater effect on wall temperature or Nusselt number compared to higher harmonics. Moreover, it is interesting to note from the expression for wall temperature that exponential damping for the first harmonic term is relatively slower. Consequently, the difference between tube wall and

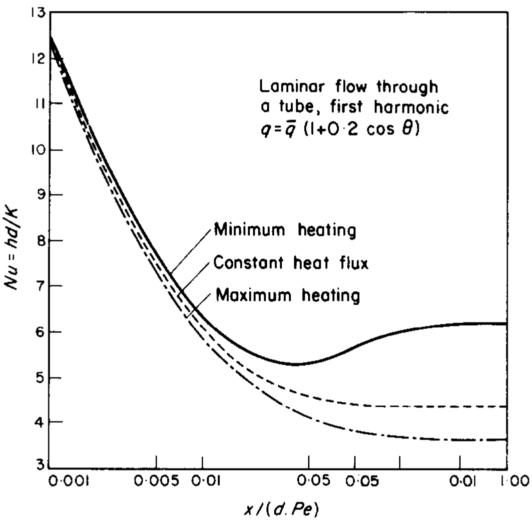

FIG. 2. Nusselt number vs. axial distance for laminar flow through a tube.

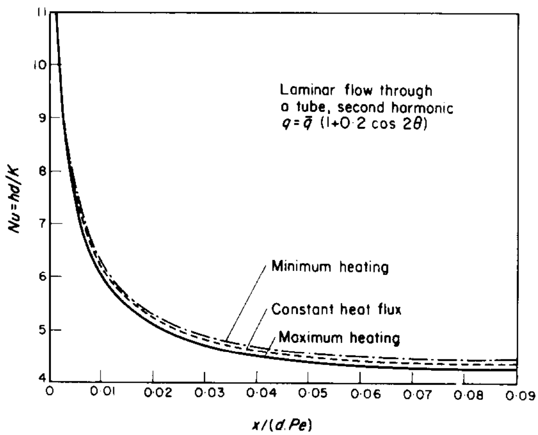

FIG. 3. Nusselt number vs. axial distance for laminar flow through a tube.

bulk mean temperature will not vary monotonically along the axial direction within some range of angular co-ordinate θ when first harmonic term is present in wall heat flux thereby indicating a rise and fall of local Nusselt number as in Fig. 2. Due to slow damping of first harmonic term the thermal entrance length for case (i) will be about three times longer than for case (ii) or for constant heat flux at the boundary.

SECOND PROBLEM: ARBITRARY WALL HEAT FLUX

The solution for an arbitrary wall heat flux distribution

$$q = \bar{q}(X)[1 + \sum_{m=1}^{\infty} \{a_m(X)\cos m\theta + b_m(X)\sin m\theta\}]$$

can be obtained in this case by applying Duhamel's superimposition theorem on the solution expressed by equation (9). The final result can be expressed in the form

$$\frac{K(t - t_i)}{a} = 4I_{00}(X) + \sum_{s=1}^{\infty} \beta_{0s}^2 c_{0s} R_{0s}(r^*, \beta_{0s}) I_{0s}(X)\, e^{-\beta_{0s}^2 X}$$
$$+ \sum_{m=1}^{\infty}\sum_{s=0}^{\infty} \beta_{ms}^2 c_{ms} R_{ms}(r^*, \beta_{ms})\, e^{-\beta_{ms}^2 X}$$
$$\{I_{ms}(X)\cos m\theta + I_{msb}(X)\sin m\theta\} \qquad (12)$$

in which

$$I_{00}(X) = \int_0^X \bar{q}(X)\,dX \qquad I_{0s}(X) = \int_0^X \bar{q}(X)\, e^{\beta_{0s}^2 X}\, dX$$

$$I_{msa}(X) = \int_0^X \bar{q}(X)\, a_m(X)\, e^{\beta_{ms}^2 X}\, dX$$

$$I_{msb}(X) = \int_0^X \bar{q}(X)\, b_m(X)\, e^{\beta_{ms}^2 X}\, dx.$$

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ON THE SOLIDIFICATION OF A WARM LIQUID FLOWING OVER A COLD WALL

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NOMENCLATURE

- a, thermal diffusion coefficient;
- C_p, specific heat;
- h, convective heat-transfer coefficient;
- k, thermal conductivity of solidified material;
- L, latent heat of fusion;
- t, temperature;
- T, dimensionless temperature;
- x, position co-ordinate;
- X, dimensionless co-ordinate;
- δ, thickness of frozen layer;
- δ_∞, thickness of frozen layer at steady state;
- Θ, modified time;
- θ, time;
- ρ, density;
- Bi, Biot number.

Subscripts
- f, at freezing temperature;
- l, liquid phase of solidifying material;
- ∞, steady state;
- w, wall.

INTRODUCTION

RECENTLY there have been some attempts by several authors [1–4] to produce a closed yet simple relation giving the freezing rate of a warm liquid. Most of the solutions available are cumbersome and involve extensive computations. Below, we give a new analytical solution which is in a closed form and easy to use in practical situations.